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Scaling Properties of $\mathbb{Z}^{k-1}$ Actions on the Circle

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We derive universal scaling properties for $\mathbb{Z}^{k-1}$ actions on the circle whose generators have rotation numbers algebraic of degree k. As for $k=2$ these properties can be explained for arbitrary k in terms of a renormalization group transformation. It has at least one trivial fixed point corresponding to an action whose generators are pure rotations. The spectrum of the linearized transformation in this fixed point is analyzed completely. The fixed point is hyperbolic with a $(k-1)$-dimensional unstable manifold. In the case $k=2$ the known results are therefore recovered.

KEY WORDS: Circle mappings; scaling behavior; algebraic rotation number; renormalization group.

1. INTRODUCTION

There have been found quite recently remarkable universal scaling properties for invertible maps of the circle S_1 whose rotation numbers are algebraic of degree $k=2$.[1-3] These properties are of some physical interest because they are closely related to a possible transition of dissipative dynamical systems from the quasiperiodic to the chaotic regime. This transition can be realized in systems with two external parameters. The main idea behind this transition is a mechanism for destroying an original smooth invariant 2-torus $\mathbb{T}_2$. Via the Poincaré map construction the problem is reduced to the behavior of circle maps with a fixed rotation number under continued iterations. It turns out that this behavior changes in an universal manner as soon as the map develops some critical point, at least when the rotation number is algebraic of degree $k=2$.

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The phenomenon which was found first in numerical experiments[1-3] can be explained by the renormalization group approach in the same spirit as Feigenbaum[4] proposed it for period doubling sequences of maps of the unit interval. In the present case this renormalization group transformation is expected to have exactly two fixed points in a suitably chosen function space. One of them is trivial and simply a pure rotation. Its existence follows from Herman's results on the conjugacy problem for smooth circle homeomorphisms[5] and it attracts all diffeomorphisms of the same rotation number. The second, nontrivial fixed point on the boundary of chaotic behavior, whose existence is very probable in the light of the numerical results in Refs. 1–3, represents a circle map with some critical point. It cannot any more be conjugated to a pure rotation by a smooth diffeomorphism, instead the conjugating map, whose existence at least for cubic critical maps was shown by Yoccoz,[6] is in general only a homeomorphism.

This picture has been shown rigorously to be true for maps with ε singularities[7] and the cubic critical case seems to be tractable also.[8]

There are two important ingredients in the theory as developed to the present day. One is Herman's result on the conjugacy problem for circle maps which holds for all rotation numbers in a set A[5] to which belong especially all algebraic numbers of degree $k=2$ and the other is the continued fraction algorithm. Lagrange's theorem tells us that the former numbers are exactly the periodic points of this algorithm (by periodicity we understand always pure and eventual periodicity). This implies that the rational approximants of these numbers fulfill very simple recursion relations with constant coefficients from which the scaling relations, at least in the diffeomorphism case, follow very easily.[3]

A natural problem then is to extend the whole theory to other rotation numbers, especially to arbitrary algebraic numbers. For them Herman's work has indeed been extended quite recently by Yoccoz.[9] He showed that all smooth diffeomorphisms of the circle whose rotation number fulfill some diophantine condition can be conjugated smoothly to a pure rotation. The famous Roth–Thue–Siegel theorem[10] shows that any algebraic number fulfills such a condition and Yoccoz' results can be applied. From this one would expect that at least the diffeomorphism case of the theory described above can be extended to such numbers. There are indeed some preliminary results in this direction.[11]

In this paper we are presenting a generalization of the theory in another direction. Instead of considering one circle map and its iterations, that means abstractly the action of the Abelian group $\mathbb{Z}$ of integers on the circle, we will study actions of the Abelian group $\mathbb{Z}^{k-1}$ for any $k \geqslant 2$ and investigate their scaling properties. We are especially interested in those actions whose generators $f_1,\ldots,f_{k-1}$ have rotation numbers algebraic of degree k. In extending the theory developed in Refs. 1–3 in this direction one is immediately lead to the problem of generalizing the continued fraction transformation to higher dimensions.

Indeed, what one really needs again is an algorithm which has both periodicity properties for higher degree algebraic numbers and defines reasonable good rational approximants for them. In the case $k=2$ both requirements are completely met by the regular continued fraction expansion, where the rational approximants are even the best possible ones.[12]

There exist several extensions of the continued fraction algorithm to higher dimensions,[13] but unfortunately none of them is as powerful as the one in dimension one.[14] These algorithms are closely related to the problem of simultaneous diophantine approximations to a set of irrational numbers, a subject much less understood than the approximation of a single irrational by rational numbers.[14] Furthermore it is known that periodicity of such an algorithm for algebraic numbers and best simultaneous approximations for them are not compatible in general.[15,16] Because periodicity is absolutely necessary at the present stage we cannot expect that we then get also best approximations as it was the case for $k=2$. But the last property is not so crucial for our purposes and reasonably good enough rational approximations do as well as we will see.

Especially for reasons of simplicity in doing analytic calculations the best algorithm for our purposes seems to be the one by Jacobi and Perron,[17,18] despite the fact that the problem of periodicity of this algorithm for arbitrary algebraic numbers is not yet settled completely.[19]

Concerning the quality of simultaneous rational approximants defined by this algorithm the situation is rather bad as pointed out already by Perron.[18] This forces us to restrict the set of rotation numbers to those whose characteristic number is of Pisot–Vijayaraghavan (P–V) type. For them the approximants as provided by the Jacobi–Perron algorithm are good enough for our purposes.

To derive universal scaling behavior for $\mathbb{Z}^{k-1}$ actions on the circle similar to the one in Refs. 1–3 for $k=2$ we follow very closely the procedure developed in the latter case. Our discussion will be limited almost entirely to the case of diffeomorphisms, which means the trivial side of the theory. If there exists as for $k=2$ also for arbitrary k a nontrivial aspect of the theory is at the moment completely open. To decide this one certainly had to perform numerical calculations as has been done for $k=2$.

In detail the paper is organized as follows: In a first section we briefly recall some of the properties of the Jacobi–Perron algorithm as far as we need them for defining the scaling relations. Most of them have been discussed already in Ref. 18. In the next section we consider $\mathbb{Z}^{k-1}$ actions on the circle which have been investigated before by Kopell[20] and show universal scaling properties for those of them whose generators have rotation numbers defining a periodic point of the Jacobi–Perron algorithm.

In a third section we derive a renormalization group transformation which explains in a simple manner the above-found scaling relations. The transformation coincides for $k = 2$ exactly with the one studied in Ref. 3. It has a trivial fixed point corresponding to generators which are pure rotations. It attracts all differentiable actions with the same rotation numbers of their generators. The spectrum of the linearization of this transformation around this trivial fixed point is analyzed completely in the last section. It turns out to be purely discrete with exactly $k - 1$ eigenvalues outside the unit disk and the rest inside it. The unstable manifold can be determined explicitly generalizing to arbitrary k the results for $k = 2$.

Because the trivial case of generators which are diffeomorphisms agrees completely with the one for $k = 2$ we expect that also for arbitrary k there exist a nontrivial aspect of the theory presented in this paper.

2. THE JACOBI–PERRON ALGORITHM

We present the algorithm here in a form introduced by Schweiger[21] as a transformation of the $(k - 1)$-dimensional unit cube $I_{k-1} \subset \mathbb{R}^{k-1}$ in complete analogy to the continued fraction transformation. If $\mathbf{x} = (x_1, ..., x_{k-1}) \in I_{k-1}$ then $T: I_{k-1} \to I_{k-1}$ is defined as

$$T\mathbf{x} = (x_2/x_1 - [x_2/x_1], ..., x_{k-1}/x_1 - [x_{k-1}/x_1], 1/x_1 - [1/x_1]) \qquad (1)$$

where $[x]$ denotes as usual the integer part of the number x. From this definition it is obvious that T can be considered a generalization of the continued fraction transformation with which it coincides in the case $k = 2$. This algorithm was set up by Jacobi[17] for $k = 3$ and extended to general k by Perron,[18] who undertook the most complete studies of its properties. The ergodic properties of this transformation have been investigated in Refs. 21 and 22 and turned out to be very similar to those of the continued fraction transformation.

The most important fact about T has been shown by Perron: any periodic point $\mathbf{x}^*$ of T has components x_i^* which necessarily all belong to one and the same algebraic number field $\mathbb{Q}[\vartheta]$ where ϑ itself is some algebraic number of degree $\leqslant k$. If furthermore the degree of ϑ is k then the x_i^*s are rationally independent and constitute together with the number 1 a basis of the field $\mathbb{Q}[\vartheta]$. It is not known however if indeed any such $(k - 1)$-tupel of algebraic numbers of degree k is a periodic point of T. In the following we will restrict our discussion always to $\mathbf{x} \in I_{k-1}$ which are periodic points of T and in fact are even fixed points. The general case can be treated in exactly the same way.

A point $\boldsymbol{\omega}^* = (\omega_1^*,..., \omega_{k-1}^*) \in I_{k-1}$ is a fixed point of T if and only if[23] there exist integers $n_1,..., n_{k-1} \in \mathbb{N} \cup \{0\}$ with $n_{k-1} \geqslant 1$ and $n_i \leqslant n_{k-1}$ for all i, such that

$$(n_{k-1}, n_{k-2},..., n_{k-j-1}) \geqslant (n_j, n_{j-1},..., n_1, 1) \tag{2}$$

for all $2 \leqslant j \leqslant k-2$ in the lexicographic order and such that

$$\boldsymbol{\omega}^* = 1/(\omega_{k-1}^* + n_{k-1})(1, \omega_1^* + n_1,..., \omega_{k-2}^* + n_{k-2}) \tag{3}$$

This means that the components ω_i^* can be expressed as

$$\omega_i^* = \omega^{*i} + n_1 \omega^{*i-1} + \cdots + n_{i-1} \omega^*, \qquad i \geqslant 2, \omega_1^* = \omega^*$$

where ω^* is the positive solution of the equation

$$\omega^k + n_1 \omega^{k-1} + \cdots + n_{k-1} \omega - 1 = 0 \tag{4}$$

For $\mathbf{n} = (n_1,..., n_{k-1}) \in \mathbb{N}^{k-1}$ fulfilling conditions (2) define a mapping $\psi_{\mathbf{n}}: I_{k-1} \to \mathbb{R}^{k-1}$ by

$$\psi_{\mathbf{n}}(\mathbf{x}) = 1/(x_{k-1} + n_{k-1})(1, x_1 + n_1,..., x_{k-2} + n_{k-2}) \tag{5}$$

This mapping can be considered a local inverse of T (Ref. 22) and it is clear that $\boldsymbol{\omega}^*$ in (3) is a fixed point of $\psi_{\mathbf{n}}$. Introducing projective coordinates $x_i = y_i/y_k$ $1 \leqslant i \leqslant k-1$ the transformation $\psi_{\mathbf{n}}$ becomes a linear transformation in $\mathbb{R}^k$ represented by the $k \times k$ matrix $A_{\mathbf{n}}$ given as

$$A_{\mathbf{n}} = \begin{bmatrix} 0 & 0 & \cdots & & \cdots & 1 \\ 1 & 0 & \cdot & & 0 & n_1 \\ 0 & 1 & 0 \cdot & \cdot & 0 & n_2 \\ \vdots & & & & & \\ 0 & \cdots & & \cdot 0 & 1 & n_{k-1} \end{bmatrix} \tag{6}$$

which is often called the companion matrix. $A_{\mathbf{n}}$ has only nonnegative entries and is even primitive as can be seen very easily. Therefore the Perron–Frobenius theorem applies in its strongest form and shows the existence of a largest positive eigenvalue λ_1 with strictly positive eigenvector $\mathbf{y}^* = (y_1^*,..., y_k^*)$ such that $\lambda_1 > |\lambda_2| \geqslant \cdots \geqslant |\lambda_k|$ for all other eigenvalues of $A_{\mathbf{n}}$. Our fixed point $\boldsymbol{\omega}^* \in I_{k-1}$ can be expressed in terms of $\mathbf{y}^*$ as $\omega_i^* = y_i^*/y_k^*$, $1 \leqslant i \leqslant k-1$. The components ω_i^* therefore belong all to the algebraic number field $\mathbb{Q}[\lambda_1]$ whose degree is $\leqslant k$ depending on the irreducibility properties of the characteristic polynomial $P_A(\lambda)$ of the matrix $A_{\mathbf{n}}$. This polynomial reads simply

$$P_A(\lambda) = (-1)^k (\lambda^k - n_{k-1}\lambda^{k-1} - \dots - n_1\lambda - 1) \tag{7}$$

This shows especially that λ_1 is a unit in $\mathbb{Q}[\lambda_1]$ that means $\lambda_1 \cdot \lambda_2 \cdot \dots \cdot \lambda_k = (-1)^{k+1}$ where $\lambda_2, \dots, \lambda_k$ denote the roots of $P_A(\lambda)$ conjugate to λ_1. Comparing expressions (7) and (4) we get a simple relation between ω^* and λ_1: $\lambda_1 = 1/\omega^* = 1/\omega_1^*$. This is a very special case of a more general result about number fields related to a periodic Jacobi–Perron expansion (see, for instance, Ref. 19, p. 109).

Consider now any k-tupel of nonnegative numbers $\mathbf{q} = (q_1, \dots, q_k) \in \mathbb{R}^k$ with $\|\mathbf{q}\| \neq 0$. It follows also from Perron–Frobenius that $\lim_{m\to\infty} \lambda_1^{-m} A_{\mathbf{n}}^m \mathbf{q} = c\mathbf{q}^*$. Specializing to integer q_is we then see that the vectors $\mathbf{q}^{(m)} = A_{\mathbf{n}}^m \mathbf{q}$ provide simultaneous diophantine approximations to the fixed point $\boldsymbol{\omega}^*$ in the sense that

$$\lim_{m\to\infty} |\omega_i^* - q_i^{(m)}/q_k^{(m)}| = 0 \tag{8}$$

for all $1 \leqslant i \leqslant k-1$, where we denoted the components of $\mathbf{q}^{(m)}$ by $q_i^{(m)}$. It follows from the work of Perron[18] that for large enough m

$$|\omega_i^* - q_i^{(m)}/q_k^{(m)}| \leqslant c\,|\lambda_2/\lambda_1|^m, \qquad 1 \leqslant i \leqslant k-1 \tag{9}$$

and that there exists at least one index j such that

$$/\omega_j^* - q_j^{(m)}/q_k^{(m)}| \geqslant c'\,|\lambda_2/\lambda_1|^m$$

for some constants c and c'. In the above expressions λ_2 denotes the second highest eigenvalue of $A_{\mathbf{n}}$ in absolute value. This shows that contrary to the case $k=2$, where λ_2 is always smaller than one in absolute value, the approximations $\mathbf{q}^{(m)}$ are in general not very good for arbitrary k, because

$$\lim_{m\to\infty} |q_k^{(m)}\omega_i^* - q_i^{(m)}| \neq 0 \tag{10}$$

in general. Because the property $\lim_{m\to\infty} q_k^{(m)}\omega_i^* - q_i^{(m)} = 0$ is absolutely crucial for deriving the scaling properties we have to restrict the discussion to those $\mathbf{n}$s which have this last property. From (9) it follows that a necessary and sufficient condition for this to hold is that $|\lambda_2| < 1$. Then all the other roots $\lambda_3, \dots, \lambda_k$ of the characteristic polynomial P_A in (7) besides the highest root λ_1 lie inside the unit disk. The number λ_1 is then called a Pisot–Vijayaraghavan (P–V) number.[10,23] This situation happens for instance for $k=3$ when λ_2 and λ_3 are complex conjugate.

In the following we are especially interested in those rational approximants $\mathbf{q}^{(m)}$ defined by the initial vector $\mathbf{q}^{(0)} = (0,..., 0, 1)$. For them we have

$$T^m(q_1^{(m)}/q_k^{(m)},..., q_{k-1}^{(m)}/q_k^{(m)}) = (0,..., 0) \tag{11}$$

for all $m \geqslant 0$. In the case $k=2$ these number $\mathbf{q}^{(m)}$ are just the generalized Fibonacci numbers defined by the recursion relation $q_i^{(m)} = q_i^{(m-2)} + nq_i^{(m-1)}$ which have been used in Ref. 3 for deriving the scaling relations.

In the case of arbitrary k we proceed as follows:

Define a transformation $T_{\mathbf{n}}$ for $\mathbf{n}$ like in (2) as

$$T_{\mathbf{n}}(\mathbf{x}) = (x_2/x_1 - n_1,..., x_{k-1}/x_1 - n_{k-2}, 1/x_1 - n_{k-1}) \tag{12}$$

It is just the Jacobi–Perron transformation T in local coordinates and defines there the inverse of the map $\psi_{\mathbf{n}}$ in (5). Therefore $T_{\mathbf{n}}\boldsymbol{\omega}^* = \boldsymbol{\omega}^*$ for the fixed point $\boldsymbol{\omega}^*$. Denoting the point $T_{\mathbf{n}}^m\mathbf{x}$ by $\mathbf{x}^{(m)}$ we can write its components $x_i^{(m)}$ as the quotient of two linear forms

$$x_i^{(m)} = \left(\sum_{j=1}^{k-1} a_{i,j}^{(m)} x_j - c_i^{(m)}\right) \Big/ - \left(\sum_{j=1}^{k-1} a_{k,j}^{(m)} x_j - c_k^{(m)}\right) \tag{13}$$

Obviously the numbers $a_{i,j}^{(m)}$ and $c_i^{(m)}$ are integers and fulfill the following recursion relations:

$$\begin{aligned} a_{i,j}^{(m+1)} &= -a_{i+1,j}^{(m)} + n_i a_{1,j}^{(m)}, \qquad 1 \leqslant i \leqslant k-2; \ 1 \leqslant j \leqslant k-1 \\ a_{k-1,j}^{(m+1)} &= a_{k,j}^{(m)} + n_{k-1} a_{1,j}^{(m)} \\ a_{k,j}^{(m+1)} &= a_{1,j}^{(m)}, \qquad 1 \leqslant j \leqslant k-1 \end{aligned} \tag{14}$$

respectively,

$$\begin{aligned} c_i^{(m+1)} &= -c_{i+1}^{(m)} + n_i c_1^{(m)} \\ c_{k-1}^{(m+1)} &= c_k^{(m)} + n_{k-1} c_1^{(m)} \\ c_k^{(m+1)} &= c_1^{(m)}, \qquad 1 \leqslant i \leqslant k-2 \end{aligned} \tag{15}$$

Relations (14) and (15) can be represented by a $k \times k$ matrix $B_{\mathbf{n}}$ whose explicit form is

$$B_{\mathbf{n}} = \begin{bmatrix} n_1 & -1 & 0 & \cdots & & & 0 \\ n_2 & 0 & -1 & 0 & \cdots & & 0 \\ \cdot & \cdot & & & & & \cdot \\ n_{k-2} & 0 & \cdots & & & -1 & 0 \\ n_{k-1} & 0 & \cdots & & & 0 & 1 \\ 1 & 0 & \cdots & & & & 0 \end{bmatrix} \tag{16}$$

The numbers $c_i^{(m)}$ are closely related to the approximation vectors $\mathbf{q}^{(m)}$ as can be seen from relation (12), which leads to

$$c_i^{(m)} = 1/q_k^{(m)} \sum_{j=1}^{k-1} \mathrm{a}_{i,j}^{(m)} q_j^{(m)}, \qquad 1 \leqslant i \leqslant k-1 \tag{17}$$

respectively,

$$c_k^{(m)} = 1/q_k^{(m-1)} \sum_{j=1}^{k-1} a_{k,j}^{(m)} q_j^{(m-1)}$$

This shows that for $1 \leqslant i \leqslant k$ the vectors $\mathbf{a}_i^{(m)} \in \mathbb{Z}^k$ defined as $(a_i^{(m)})_j = a_{i,j}^{(m)}$ for $1 \leqslant j \leqslant k-1$ and $(a_i^{(m)})_k = c_i^{(m)}$ define Diophantine approximations to the linear form

$$l(\mathbf{x}) = \sum_{j=1}^{k-1} \omega_j^* x_j - x_k \tag{18}$$

For this we have to show that for all $1 \leqslant i \leqslant k$ we have

$$\lim_{m \to \infty} |l(\mathbf{a}_i^{(m)})| = 0$$

But $\boldsymbol{\omega}^*$ being a fixed point of $T_{\mathbf{n}}$ we have trivially for all m and all $1 \leqslant i \leqslant k-1$

$$\sum_{j=1}^{k-1} a_{i,j}^{(m)} \omega_j^* - c_i^{(m)} = -\omega_i^* \left(\sum_{j=1}^{k-1} a_{k,j}^{(m)} \omega_j^* - c_k^{(m)} \right) \tag{19}$$

We therefore have only to show that

$$\lim_{m \to \infty} |l(\mathbf{a}_k^{(m)})| = 0$$

Using expression (17) we can write this as

$$l(\mathbf{a}_k^{(m)}) = 1/q_k^{(m-1)} \sum_{j=1}^{k-1} a_{k,j}^{(m)} (q_k^{(m-1)} \omega_j^* - q_j^{(m-1)})$$

To find the large-m behavior of $l(\mathbf{a}_k^{(m)})$ we need to know therefore the behavior of the numbers $a_{k,j}^{(m)}$ for large m. This is obviously determined by the spectral properties of the matrix $B_{\mathbf{n}}$ in (16) because $a_{k,j}^{(m)} = (B_{\mathbf{n}}^m \mathbf{a}_k^{(0)})_j$.

The characteristic polynomial of $B_{\mathbf{n}}$ can be calculated explicitly to give

$$P_B(\lambda) = (-\lambda)^k + n_1(-\lambda)^{k-1} + \cdots + n_{k-1}(-\lambda) - 1 \tag{20}$$

Comparing this with the characteristic polynomial P_A of A_n in (7) then shows that the spectrum $\sigma(B_n)$ is just given as

$$\rho \in \sigma(B_n) \Leftrightarrow -\rho^{-1} \in \sigma(A_n) \tag{21}$$

The highest eigenvalue in absolute value of B_n which determines in general the behavior of $a_{i,j}^{(m)}$ is therefore the eigenvalue $\rho_1 = -1/\lambda_k$. This together with Perron's results (9) concerning the approximation vector $\mathbf{q}^{(m)}$ shows that for large m

$$|l(\mathbf{a}_k^{(m)})| \leqslant c\,|\lambda_2^2 \cdot \lambda_3 \cdot \ldots \cdot \lambda_{k-1}|^m, \qquad k \geqslant 4 \tag{22}$$

respectively,

$$|l(\mathbf{a}_k^{(m)})| \leqslant c\,|\lambda_2 \cdot \lambda_3|^m \qquad \text{for} \quad k = 3$$

Because λ_1 was by assumption a P–V number the right-hand side vanishes exponentially fast with increasing m. The above estimates show also that the approximation vectors $\mathbf{a}_k^{(m)}$ for the form l are also not very good in general. Only in the case $k = 3$ and for complex conjugates roots λ_2, λ_3 we get the best approximations[15]

$$|l(\mathbf{a}^{(m)})| \leqslant c/\max_{1 \leqslant j \leqslant k} |a_j^{(m)}|^2$$

In the general case one gets only

$$|l(\mathbf{a}^{(m)})| \leqslant c/\max_{1 \leqslant j \leqslant k} |a_j^{(m)}|^\alpha$$

where α is determined by $|\lambda_2^2 \cdot \lambda_3 \cdot \ldots \cdot \lambda_{k-1}| = O(|\lambda_k|^\alpha)$ and hence can be rather small. Nevertheless property (22) will allow us to derive universal scaling behavior for $\mathbb{Z}^{k-1}$ actions on the circle.

3. SCALING PROPERTIES OF $\mathbb{Z}^{k-1}$ ACTIONS ON THE CIRCLE

A $\mathbb{Z}^{k-1}$ action on the circle is defined by a homomorphism φ of the Abelian group $\mathbb{Z}^{k-1}$ into the group of all orientation-preserving diffeomorphisms $\mathrm{Diff}(S_1)$ of the circle S_1, such that the induced mapping $\tilde{\varphi}: \mathbb{Z}^{k-1} \times S_1 \to S_1$ defined by $\tilde{\varphi}(g, x) = \varphi(g)x$ is differentiable. Such an action is obviously completely determined by its generators $f_1, \ldots, f_{k-1} \in \mathrm{Diff}(S_1)$ where $f_i = \varphi(g_i)$ and the g_i, $1 \leqslant i \leqslant k-1$, generate the group $\mathbb{Z}^{k-1}$. The differentiable actions of any noncompact abelian group on S_1 have been discussed by Kopell.[20] Because the generators $f_1, \ldots, f_{k-1}$ commute they all belong to the intersection $\bigcap_{i=1}^{k-1} C(f_i)$ where $C(f_i)$ denotes the centralizer of

f_i which is just $\{g \in \mathrm{Diff}(S_1): f \circ g = g \circ f_i\}$. It was shown in Ref. 20 that $C(f) \cong SO(2)$ if the rotation number ρ of f is irrational and f is diffeomorphic conjugate to the pure rotation $R_{\rho(f)}$. In this case $C(f)$ is simply $\{g = h \circ R_\alpha \circ h^{-1}, \alpha \in (0, 2\pi]\}$, where $f = h \circ R_{\rho(f)} \circ h^{-1}$. Of special interest for us are those actions whose generators f_i have rotation numbers $\rho(f_i)$ which define a fixed point $\boldsymbol{\omega}^*$ of the Jacobi–Perron transformation T and are therefore algebraic of degree k. From the results of Kopell and Yoccoz it then follows that $f_i = h \circ R_{\omega_i^*} \circ h^{-1}$ for all $1 \leqslant i \leqslant k-1$.

As usual we work in the following with a lift $\tilde{f}: \mathbb{R} \to \mathbb{R}$ of any circle homeomorphism $f: S_1 \to S_1$, that means a monotonic increasing homeomorphism of the real line with the property

$$\tilde{f}(x+1) = \tilde{f}(x) + 1 \tag{23}$$

It is clear that such a lift is determined by f only up to an integer constant but what we are going to do does not depend on this constant and defines therefore properties of f itself. We will therefore identify from now on f and the lift $\tilde{f}$. To define then the scaling relations for an $\mathbb{Z}^{k-1}$ action on S_1 with generators $f_1, \ldots, f_{k-1}$ we set for $m \geqslant 1$

$$\lambda_{(m-1)} := -(f_1^{a_{k,1}^{(m)}} \circ \cdots \circ f_{k-1}^{a_{k,k-1}^{(m)}}(0) - c_k^{(m)}) \tag{24}$$

where the integers $a_{k,j}^{(m)}$ respectively $c_k^{(m)}$ are defined in (14) and (15) with initial values

$$\begin{aligned} &a_{i,j}^{(0)} = \delta_{i,j}, \qquad 1 \leqslant i \leqslant k, \quad 1 \leqslant j \leqslant k-1 \\ &c_i^{(0)} = 0, \quad 1 \leqslant i \leqslant k-1; \qquad c_k^{(0)} = 1 \end{aligned} \tag{25}$$

Furthermore define the functions $\xi_i^{(m)}(x)$ for $1 \leqslant i \leqslant k$ as

$$\xi_i^{(m)}(x) := 1/\lambda_{(m-1)}(f_1^{a_{i,1}^{(m)}} \circ \cdots \circ f_{k-1}^{a_{i,k-1}^{(m)}}(\lambda_{(m-1)}x) - c_i^{(m)}) \tag{26}$$

Then obviously

$$\xi_k^{(m)}(0) = -1 \tag{27}$$

for all $m \geqslant 1$. Using the bound (22) one shows easily that

$$|\lambda_{(m)}| \leqslant c\,|\lambda_2^2 \cdot \lambda_3 \cdot \ldots \cdot \lambda_{k-1}|^m \tag{28}$$

where the constant c depends only on the diffeomorphism h. By exactly the arguments used in Ref. 3 it then follows that for all $\mathbb{Z}^{k-1}$ actions on S_1 with generators $f_i = h \circ R_{\omega_i^*} \circ h^{-1}$ the following limits exist and do not depend on h:

$$\lim_{m\to\infty} \xi_i^{(m)}(x) = R_{\omega_i^*}(x), \qquad 1 \leqslant i \leqslant k-1$$
$$\lim_{m\to\infty} \xi_k^{(m)}(x) = x - 1 \tag{29}$$

From relations (14) and (19) it also follows furthermore that

$$\lim_{m\to\infty} (\lambda_{(m)}/\lambda_{(m-1)}) = -\omega_1^* = -1/\lambda_1 \tag{30}$$

independent of h.

Relations (29) and (30) define universal scaling behavior for $\mathbb{Z}^{k-1}$ actions on the circle. In the case $k=2$ these scaling relations coincide exactly with the ones found in Refs. 1–3.

To see this one only has to convince oneself that in this case the numbers $a_{1,1}^{(m)}$ and $a_{2,1}^{(m)}$ are exactly the best approximants $q_2^{(m)}$ and $q_2^{(m-1)}$ defined by the continued fraction transformation and similarly $c_1^{(m)} = q_1^{(m-1)}$. In the case $k=2$ the matrices A_n and B_n in (6), respectively (16), have the form

$$A_n = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B_n = \begin{bmatrix} n & 1 \\ 1 & 0 \end{bmatrix}$$

and generate indeed the best approximants.

In complete analogy to the case $k=2$ we can ask also for arbitrary k for further scaling relations. Remember, that in the former case for instance the numbers $\tau^{(m)}$ defined by the condition $(R_{\tau_{(m)}} \circ f)^{q_2^{(m)}}(0) = q_1^{(m)}$ scaled like $\lim_{m\to\infty} [\tau^{(m)}/\tau^{(m-1)}] = -\omega_1^*$.[3] In our general case of arbitrary k one would then expect that also the numbers $\tau_i^{(m)}$ defined by the conditions

$$(R_{\tau_1^{(m)}} \circ f_1)^{a_{i,1}^{(m)}} \circ \dots \circ (R_{\tau_{k-1}^{(m)}} \circ f_{k-1})^{a_{i,k-1}^{(m)}}(0) = c_i^{(m)} \tag{31}$$

$1 \leqslant i \leqslant k-1$, should also show some scaling behavior. This is however not the case in general as we will show now. Consider for this the special case $f_i = R_{\omega_i^*}$. Then one can solve the equations (31) for the $\tau_i^{(m)}$ and gets $\sum_{j=1}^{k-1} a_{i,j}^{(m)}(\omega_j^* + \tau_j^{(m)}) = c_i^{(m)}$ with the solution

$$\tau_j^{(m)} = -\omega_j^* + q_j^{(m)}/q_k^{(m)}$$

Therefore we get for the ratio $\tau_j^{(m)}/\tau_j^{(m-1)}$, the expression

$$\tau_j^{(m)}/\tau_j^{(m-1)} = q_k^{(m-1)}/q_k^{(m)} \cdot (q_k^{(m)}\omega_j^* - q_j^{(m)})/(q_k^{(m-1)}\omega_j^* - q_j^{(m-1)})$$

The behavior of the term $q_k^{(m)}\omega_j^* - q_j^{(m)}$ for large m is determined by one of the eigenvalues $\lambda_2,..., \lambda_k$ of $A_{\mathbf{n}}$ depending on j. If this eigenvalue, however, is complex then also the complex conjugate eigenvalue contributes above and the limit does not exist and is oscillating instead. The limit exists on the other hand if the relevant eigenvalue is real and we get a scaling law

$$\lim_{m\to\infty} \tau_j^{(m)}/\tau_j^{(m-1)} = \lambda^{(j)}/\lambda_1 \tag{32}$$

where $\lambda^{(j)}$ is the relevant eigenvalue. This shows that depending on j we get either a limit or the limit does not exist. In the case $k=2$ there exists only the eigenvalue λ_2 which moreover is always real. For $k=3$ both situations can already appear: either λ_2 is complex and the limit does not exist or λ_2 and λ_3 are real and give rise to two different values for the limit in (32). This happens for instance for $\mathbf{n} = (9, 15)$. We will see next how this is reflected in the fixed point structure of the renormalization group transformation belonging to these scaling relations. Let us remark that a standard argument shows that the above properties hold for all h.

4. A RENORMALIZATION GROUP TRANSFORMATION FOR ARBITRARY *k*

To derive this transformation we follow the standard procedure as used for instance in Ref. 3. Take the functions $\xi_i^{(m)}$ defined in (26) and look for a recursion relation in m. Using relations (14) and (15) one sees easily that every $\xi_i^{(m+1)}$ can be rewritten as a functional of the $\{\xi_j^{(m)}\}$, namely,

$$\begin{aligned}
\xi_i^{(m+1)}(x) &= \lambda_{(m-1)}/\lambda_{(m)}(\xi_1^{(m)})^{n_i} \circ (\xi_{i+1}^{(m)})^{-1}(\lambda_{(m)}/\lambda_{(m-1)}x), \qquad 1 \leqslant i \leqslant k-2 \\
\xi_{k-1}^{(m+1)}(x) &= \lambda_{(m-1)}/\lambda_{(m)}(\xi_1^{(m)})^{n_{k-1}} \circ \xi_k^{(m)}(\lambda_{(m)}/\lambda_{(m-1)}x) \\
\xi_k^{(m+1)}(x) &= \lambda_{(m-1)}/\lambda_{(m)}\xi_1(\lambda_{(m)}/\lambda_{(m-1)}x)
\end{aligned} \tag{33}$$

where $\xi_k^{(m)}(0) = -1$ for all m.

Performing the limit $m\to\infty$ in (33) by remembering that at least for certain f_i it really exists we get a transformation $\mathscr{R}$ as

$$\mathscr{R}\xi(x) = \begin{cases} (1/\lambda)\, \xi_1^{n_i} \circ \xi_{i+1}^{-1}(\lambda x), & 1 \leqslant i \leqslant k-2 \\ (1/\lambda)\, \xi_1^{n_{k-1}} \circ \xi_k(\lambda x) \\ (1/\lambda)\, \xi_1(\lambda x) \end{cases} \tag{34}$$

with

$$\lambda = -\xi_1(0)$$

In the case $k=2$ the transformation $\mathscr{R}$ is just a little modification of the one used in Ref. 3:

$$\mathscr{R}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}(x)=\frac{1}{\lambda}\begin{pmatrix}\xi_1^n\circ\xi_2(\lambda x)\\ \xi_1(\lambda x)\end{pmatrix},\qquad \lambda=-\xi_1(0)$$

We see that the transformation $\mathscr{R}$ acts in a space of k-tupels of functions $\xi_i(x)$ on the real line. In analogy to the case $k=2$ one would like to interpret the functions $\boldsymbol{\xi}=(\xi_1,\dots,\xi_k)$ if possible as a $(k-1)$-tupel of circle mappings. It turns out that this can be done indeed. To give the construction let us consider the space $D_\mathbf{n}$ of k-tupels $\boldsymbol{\xi}=(\xi_1,\dots,\xi_k)$ of monotone increasing diffeomorphisms ξ_i of the real line such that

$$\begin{array}{lll}
\text{(D1)} & \xi_k(0)=-1, \qquad \xi_i\circ\xi_k(0)<0<\xi_i(0), & 1\leqslant i\leqslant k-1\\
\text{(D2)} & \xi_i\circ\xi_k(0)=\xi_k\circ\xi_i(0), & 1\leqslant i\leqslant k-1\\
\text{(D3)} & \xi_1^{n_i}\circ\xi_{i+1}^{-1}(0)<0<\xi_1^{n_i+1}\circ\xi_{i+1}^{-1}(0), & \xi_1^{n_{k-1}}(-1)<0<\xi_1^{n_{k-1}+1}(-1)\\
\text{(D4)} & \xi_1\circ\xi_{i+1}^{-1}(0)=\xi_{i+1}^{-1}\circ\xi_1(0) &
\end{array}\tag{35}$$

for all $1\leqslant i\leqslant k-2$

Properties (35) allow us to associate to every $\xi\in D_\mathbf{n}$ a $(k-1)$ tupel $\mathbf{f}^\xi=(f_1^\xi,\dots,f_{k-1}^\xi)$ of circle homeomorphisms and a rotation vector $\boldsymbol{\rho}(\xi)=(\rho(f_1^\xi),\dots,\rho(f_{k-1}^\xi))$ in the following way:

Define

$$f_i^\xi(x):=\begin{cases}\xi_i(x), & \xi_i\circ\xi_k(0)\leqslant x\leqslant 0\\ \xi_i\circ\xi_k(x), & 0\leqslant x=\xi_i(0)\end{cases}\tag{36}$$

Because of (D1) the interval $[\xi_i\circ\xi_k(0),\xi_i(0)]$ is not empty and contains the point 0 in its interior. Property (D2) shows that $f_i^\xi(\xi_i\circ\xi_k(0))=f_i^\xi(\xi_i(0))$. By identifying therefore the two endpoints of the interval $[\xi_i\circ\xi_k(0),\xi_i(0)]$ we get a homeomorphism of the circle to which we can associate also a rotation number $\rho(f_i^\xi)$. The circle homeomorphisms $f_i\colon S_1\to S_1$ can be embedded trivially into the space D_n in defining $\xi_i^f(x)=f_i(x)$ for $1\leqslant i\leqslant k-1$ and $\xi_k^f(x)=x-1$. To be precise the rotation numbers $\rho(f_i)$ must also fulfill certain conditions so that properties (D1)–(D4) are true. It is clear that $f_i^{\xi^f}(x)=f_i(x)$ in that case and therefore $\boldsymbol{\rho}(\xi^f)=\boldsymbol{\rho}(f)$. Now consider the special case where $\mathbf{f}=(f_1,\dots,f_{k-1})$ is given by $f_i=h\circ R_{\omega_i^*}\circ h^{-1}$. Then a simple calculation shows that

$$\mathscr{R}^m\boldsymbol{\xi}^f=\boldsymbol{\xi}^{(m)}\tag{37}$$

where the vector $\boldsymbol{\xi}^{(m)}$ has been defined in (26). This shows that the transformation $\mathscr{R}$ in (34) reproduces exactly the iterations on the $(k-1)$-tupel

$(f_1,..., f_{k-1})$ we are interested in. This remark also shows that the point $\boldsymbol{\xi}^*(x) = (R_{\omega_1^*}(x),..., R_{\omega_{k-1}^*}(x), x-1)$ is a fixed point of $\mathscr{R}$ and that all points $\boldsymbol{\xi}(x) = (h \circ R_{\omega_1^*} \circ h^{-1}(x),..., h \circ R_{\omega_{k-1}^*} \circ h^{-1}(x), x-1)$ belong to the stable manifold of this fixed point.

A straightforward calculation shows that conditions (D3) and (D4) make sure that $\mathscr{R}\boldsymbol{\xi}$ fulfills conditions (D1) and (D2) for any $\boldsymbol{\xi} \in D_n$ and posseses therefore a rotation vector $\boldsymbol{\rho}(\mathscr{R}\xi)$. What is the relation between $\boldsymbol{\rho}(\xi)$ and $\boldsymbol{\rho}(\mathscr{R}\xi)$. Let us consider the case where $\boldsymbol{\xi}(x) = (R_{\omega_1}(x),..., R_{\omega_{k-1}}(x), x-1) \in D_{\mathbf{n}}$. Condition (D1) simply says that $0 < \omega_i < 1$ for all $1 \leqslant i \leqslant k-1$. Condition (D3) on the other hand tells us that $n_{k-1} < 1/\omega_1 < n_{k-1}+1$ and $n_i < \omega_{i+1}/\omega_1 < n_i + 1$ for $1 \leqslant i \leqslant k-2$.

For $\mathscr{R}\boldsymbol{\xi}(x)$ we get

$$\begin{aligned} \mathscr{R}\boldsymbol{\xi}_i(x) &= x + \omega_{i+1}/\omega_1 - n_i, \qquad \text{for} \quad 1 \leqslant i \leqslant k-2 \\ \mathscr{R}\boldsymbol{\xi}_{k-1}(x) &= x + 1/\omega_1 - n_{k-1} \\ \mathscr{R}\boldsymbol{\xi}_k(x) &= x - 1 \end{aligned}$$

But this shows that $\boldsymbol{\rho}(\mathscr{R}\xi) = (\omega_2/\omega_1 - n_1,..., \omega_{k-1}/\omega_1 - n_{k-2}, 1/\omega_1 - n_{k-1})$ or

$$\begin{aligned} \boldsymbol{\rho}(\mathscr{R}\xi) = (&\rho(f_2^\xi)/\rho(f_1^\xi) - n_1,..., \rho(f_{k-1}^\xi)/\rho(f_1^\xi) \\ &- n_{k-2}, 1/\rho(f_1^\xi) - n_{k-1}) \end{aligned} \tag{38}$$

This discussion also shows that the $(k-1)$-dimensional manifold $W^u(\boldsymbol{\xi}^*) = \{\boldsymbol{\xi}: \xi_i(x) = R_{\omega_i}(x), \xi_k(x) = x-1: \omega_i \in \mathbb{R}\}$ is left invariant under the transformation $\mathscr{R}$. We will see immediately that $W^u(\boldsymbol{\xi}^*) \cap D_n$ is just the unstable manifold of the fixed point $\boldsymbol{\xi}^*$ in $D_{\mathbf{n}}$. To see this we have to investigate the spectrum of the linearization $D\mathscr{R}(\boldsymbol{\xi}^*)$ of the transformation $\mathscr{R}$ in the point $\boldsymbol{\xi}^*$. It turns out that exactly the same technique can thereby be applied as it was used in Ref. 3 for the special case $k=2$.

In a first step one simplifies the transformation $\mathscr{R}$ in (34) in such a way that the rescaling factor λ becomes constant. For this purpose one defines two auxiliary transformations $\mathscr{R}_1$ and $\mathscr{R}_2$ in suitably chosen spaces as follows:

$$\mathscr{R}_1\boldsymbol{\xi}(x) = (1/\tilde{\lambda})\, \boldsymbol{\xi}(\tilde{\lambda}x) \tag{39}$$

with $\tilde{\lambda} = -\xi_k(0)$ and

$$\mathscr{R}_2\boldsymbol{\xi}(x) = (1/\lambda^*) \begin{cases} \xi_1^{n_i} \circ \xi_{i+1}^{-1}(\lambda^* x), & 1 = i = k-2 \\ \xi_1^{n_{k-1}} \circ \xi_k(\lambda^* x) \\ \xi_1(\lambda^* x) \end{cases} \tag{40}$$

with $\lambda^* = -\xi_1^*(0) = -\omega_1^*$.

As in Ref. 3 one shows that $\mathscr{R} = \mathscr{R}_1 \circ \mathscr{R}_2 = \mathscr{R}_1 \circ \mathscr{R}_2 \circ \mathscr{R}_1$ and $\mathscr{R}_1^2 = \mathscr{R}_1$. Furthermore $\mathscr{R}_1 \boldsymbol{\xi}^* = \mathscr{R}_2 \boldsymbol{\xi}^* = \boldsymbol{\xi}^*$. As for $k=2$ it turns out that any eigenvalue of $D\mathscr{R}(\boldsymbol{\xi}^*)$ is at the same time an eigenvalue with exactly the same multiplicity of $D\mathscr{R}_2(\boldsymbol{\xi}^*)$ and any eigenvalue $\lambda \neq 1$ of $D\mathscr{R}_2(\boldsymbol{\xi}^*)$ is an eigenvalue of $D\mathscr{R}(\boldsymbol{\xi}^*)$. Therefore the spectrum of $D\mathscr{R}_2(\boldsymbol{\xi}^*)$ determines the one of the operator $D\mathscr{R}(\boldsymbol{\xi}^*)$ we are looking for. For the sake of simplicity we restrict our discussion from now on to the case $k=3$ but mention that all arguments can be carried through for arbitrary k, the formulas get only slightly longer. The derivatives of the transformations $\mathscr{R}_1$ and $\mathscr{R}_2$ in the fixed point $\boldsymbol{\xi}^*$ can easily be calculated:

$$(D\mathscr{R}_1(\boldsymbol{\xi}^*)\,\mathbf{h})(x) = h_3(0)\,\boldsymbol{\omega} + \mathbf{h}(x) \tag{41}$$

where $\boldsymbol{\omega} = (\omega_1^*, \omega_2^*, -1)$, respectively,

$$(D\mathscr{R}_2(\boldsymbol{\xi}^*)\,\mathbf{g})(x) = (1/\lambda^*)\begin{pmatrix} -g_2(\lambda^* x - \omega_2^*) + \sum_{j=0}^{n_1-1} g_1(\lambda^* x - \omega_2^* + j\omega_1^*) \\ g_3(\lambda^* x) + \sum_{j=0}^{n_2-1} g_1(\lambda^* x - 1 + j\omega_1^*) \\ g_1(\lambda^* x) \end{pmatrix} \tag{42}$$

The operator $D\mathscr{R}_2(\boldsymbol{\xi}^*)$ can be treated very easily; in fact it is a kind of composition operator which we discussed some time ago in Ref. 24. When restricted to a certain B space of holomorphic functions it defines a nuclear operator with very simple spectral properties. To determine its spectrum which obviously consists only of eigenvalues one writes down the eigenvalue equation for $g_1(x)$. This gives

$$\begin{aligned} g_1(x) = (1/\lambda^*) &\sum_{j=0}^{n_1-1} g_1(\lambda^* x - \omega_2^* + j\omega_1^*) - 1/(\lambda\lambda^{*2}) \\ &\times \sum_{j=0}^{n_2-1} g_1(\lambda^{*2} x - \lambda^*\omega_2^* - 1 + j\omega_1^*) - 1/(\lambda^2\lambda^{*3})\, g_1(\lambda^{*3} x - \lambda^{*2}\omega_2^*) \end{aligned} \tag{43}$$

But this can be solved by the ansatz $g_1(x) = \sum_{j=0}^{N} a_j x^j$ which leads to the eigenvalue equation

$$\lambda = \lambda^{*N-1} n_1 - 1/\lambda\; \lambda^{*2(N-1)} n_2 - 1/\lambda^2\; \lambda^{*3(N-1)}$$

Setting therefore $\lambda = \lambda^{*N-1}\lambda'$ we get

$$\lambda' = n_1 - (1/\lambda')\, n_2 - 1/\lambda'^2 \qquad \text{or} \qquad \lambda'^3 - n_1\lambda'^2 + n_2\lambda' + 1 = 0$$

Comparing this with equation (4) we see that the solutions of the above equation are exactly $-\omega_1^*, -\omega', -\omega''$ where ω', ω'' are the conjugates of the algebraic number $\omega_1^* = \omega^*$. Therefore the spectrum $\sigma(D\mathscr{R}_2(\xi^*))$ is given by the numbers

$$\{(-1)^n \omega^{*n-1}\omega^*, (-1)^n\omega^{*n-1}\omega', (-1)^n\omega^{*n-1}\omega'' : n \in \mathbb{N} \cup \{0\}\} \quad (44)$$

Only five of these numbers are bigger than one in absolute value: $1, \omega^{*-1}\omega'$, $\omega^{*-1}\omega'', -\omega', -\omega''$. Because $|\omega^*\omega'\omega''| = 1$ we have in fact $|\omega^*\omega'| < 1$, $|\omega^*\omega''| < 1$.

It turns out that only two of the above numbers are really eigenvalues of the operator $D\mathscr{R}(\xi^*)$ when $\mathscr{R}$ is considered as operating in the space $D_{\mathbf{n}}$. This means namely that the eigenfunctions have to be tangent to $D_{\mathbf{n}}$. If we denote the tangent vectors to the manifold $D_{\mathbf{n}}$ by $\mathbf{g}(x) = (g_1(x),..., g_k(x))$ then they fulfill the conditions (at the fixed point ξ^*):

$$\begin{aligned} &g_k(0) = 0 \\ &g_k(0) + g_i(-1) = g_k(\omega_i^*) + g_i(0), && 1 \leqslant i \leqslant k-1 \quad (44) \\ &g_1(-\omega_{i+1}^*) - g_{i+1}(-\omega_{i+1}^*) = g_1(0) - g_{i+1}(\omega_1^* - \omega_{i+1}^*), && 1 \leqslant i \leqslant k-2 \end{aligned}$$

Let us restrict our discussion from now on again to the case $k = 3$. The case of arbitrary k can be handled in exactly the same way. If λ is one of the eigenvalues $1, \omega'/\omega_1^*, \omega''/\omega_1^*$ of $D\mathscr{R}_2(\xi^*)$ the corresponding eigenfunction has the form

$$\mathbf{g}_\lambda(x) = \begin{pmatrix} 1 \\ n_1 - \lambda\lambda^* \\ 1/(\lambda\lambda^*) \end{pmatrix} \quad (45)$$

The corresponding eigenfunction $\mathbf{h}_2$ to exactly the same eigenvalue for the operator $D\mathscr{R}(\xi^*)$ can be obtained simply by applying the operator $D\mathscr{R}_1(\xi^*)$ to $\mathbf{g}_\lambda$. This gives

$$\mathbf{h}_\lambda(x) = \begin{pmatrix} 1 - 1/\lambda \\ n_1(1 - 1/\lambda) + \omega_1^*(1 - 1/\lambda^2) \\ 0 \end{pmatrix} \quad (46)$$

For $\lambda = 1$ we see that $\mathbf{h}_1(x) = \mathbf{0}$ and $\lambda = 1$ is not an eigenvalue of $D\mathscr{R}(\xi^*)$. On the other hand for $\lambda = \omega'/\omega^*$ or $\lambda = \omega''/\omega^*$ the functions $\mathbf{h}_\lambda(x)$ turn out to be indeed eigenfunctions of $D\mathscr{R}(\xi^*)$ tangent to the space $D_{\mathbf{n}}$ in the point ξ^* because obviously conditions (44) are then satisfied. In a next step one has to determine in complete analogy the eigenfunctions corresponding to the eigenvalues $\lambda = -\omega'$, respectively, $\lambda = -\omega''$. In this case the eigenfunction $\mathbf{g}_2(x)$ has the form

$$\mathbf{g}_\lambda(x) = \begin{pmatrix} x + \alpha \\ x(1/\lambda^2 - 1/\lambda) + \alpha(1/(\lambda\lambda^*)^2 + n_2/\lambda) - n_2/\lambda \\ \quad + 1/(2\lambda^*)\, n_2(n_2 - 1) \\ (1/\lambda)\, x + \alpha/(\lambda\lambda^*) \end{pmatrix}$$

where α is determined uniquely by Eq. (43). The corresponding eigenvectors $\mathbf{h}_\lambda$ of $D\mathscr{R}(\boldsymbol{\xi}^*)$ then read as follows:

$$\mathbf{h}_\lambda(x) = \begin{pmatrix} x + \alpha + \omega_1^* \alpha/(\lambda\lambda^*) \\ x(1/\lambda^2 - 1/\lambda) + \omega_2^* \alpha/(\lambda\lambda^*) + \alpha(1/(\lambda\lambda^*)^2 + n_2/\lambda) \\ \quad + \omega_1^*/(2\lambda)\, n_2(n_2 - 1) \\ (1/\lambda)\, x \end{pmatrix}$$

A straightforward calculation shows that conditions (44) are not satisfied by these $\mathbf{h}_\lambda$. Therefore the eigenvalues $\lambda = -\omega'$ and $\lambda = -\omega''$ do not belong to $\sigma(D\mathscr{R}(\boldsymbol{\xi}^*))$ when the operator $\mathscr{R}$ is considered as acting in the space $D_{\mathbf{n}}$. But this shows that only the eigenvalues ω'/ω_1^* and ω''/ω_1^* lie outside the unit circle and lead therefore to a two-dimensional unstable manifold $W^u(\boldsymbol{\xi}^*)$ of the trivial fixed point $\boldsymbol{\xi}^*$. This is exactly what we expected from the discussion of the scaling behavior of the numbers $\tau_i^{(m)}$ in (32). This result for $k = 3$ can now immediately be extended to arbitrary degree k. The unstable manifold of $\boldsymbol{\xi}^*$ has there dimension $k - 1$ and is explicitly given by

$$W^u(\boldsymbol{\xi}^*) = \{\boldsymbol{\xi} \in D_{\mathbf{n}}: \xi_k(x) = x - 1, \xi_i(x) = x + c_i, 1 \leqslant i \leqslant k - 1, c_i \in \mathbb{R}\}$$

The case $k = 2$ therefore fits perfectly into our general theory.

Remarks. The above results for the trivial fixed point $\boldsymbol{\xi}^*$ can presumably be made completely rigorous in exactly the way done for $k = 2$ in Ref. 7. Concerning the existence of a second nontrivial fixed point a little problem arises through the fact that the action of $\mathscr{R}$ involves the inverse functions $\xi_i^{-1}(x)$. As soon as there is a critical point in one of the functions ξ_i its inverse is not anymore everywhere differentiable and the operator $\mathscr{R}$ leads immediately outside the space $D_{\mathbf{n}}$ which contains only k-tupels of differentiable functions. A second nontrivial fixed point of $\mathscr{R}$ can therefore only exist in a space of functions which are less regular. To get some feeling in this direction, however, one should do computer calculations in iterating some $(k - 1)$-tupel of functions $(f_1, ..., f_{k-1})$ with $f_i = h \circ R_{\omega_i^*} \circ h^{-1}$, where h is not a diffeomorphism. If f_1 for instance has rotation number $\rho(f_1) = \omega_1^*$

and one cubic critical point then there exists such a homeomorphism conjugating it to the pure rotation $R_{\omega_1^*}$ by a recent result of Yoccoz.[11]

Another interesting problem would be to extend also the work of Manton and Nauenberg[25,26] to $\mathbb{Z}^{k-1}$ actions in the complex plane in the same way we did here for such actions on the circle.

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